Symplectic Integrators applied to Beam Dynamics in Circular Accelerators
Basis for Constructing Integration Schemes for Hamiltonian Systems

Laurent S. Nadolski
Synchrotron SOLEIL - Beam Dynamics Group

nadolski@synchrotron-soleil.fr

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Outline

Introduction
  Overview of course

Numerical integration

Integration Methods

Symplectic Integration ab Ovo

Refresher on Hamiltonian Mechanics

Application to Single Element

Conclusion

References
Overview of the course

1. Understand what is numerical integration and especially the use classes of symplectic integrators.
2. Basic schemes of integration.
3. How to build integrators to an arbitrary order.
4. Integration of simple elements as straight sections, dipole, quadrupole, and sextupole magnets.

These methods can be applied in many other contexts

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Definition

Let us start with a dynamical system of \textbf{n degrees of freedom} ruled by system of differential equations,

\[
\frac{dx_k}{dt} = f(x, t), \quad k = 1 \ldots n \tag{1}
\]

What are the solutions \(x_k(t)\)?

Case of a pendulum, \(n = 2\)

\[
H(q, p) = \frac{p^2}{2} + \cos q \tag{2}
\]

\[
\begin{cases}
\frac{dq}{dt} = \frac{\partial H}{\partial q} = p \\
\frac{dp}{dt} = -\frac{\partial H}{\partial p} = \sin q.
\end{cases} \tag{3}
\]

\textbf{Objective}: find a numerical method for integrating these equations and study the dynamics of the system (convergence, fixed point, stability, resonance, etc.).

Definition (Con’t)

Integration of the equations of the motion is often referred as \textbf{Tracking code} in Accelerator Physics.

Let us study a \textbf{circular accelerator} made of many magnets (dipole, quadrupole, and sextupole magnets)

1. Define a \textbf{global Hamiltonian} \(\rightarrow\) complicated and not so practical
2. Define a \textbf{local Hamiltonian} for each individual element with \textbf{its local reference frame} (curvilinear, rectangular, ...) determined by \textbf{its geometry} (shape, symmetry of the magnetic-field) \(\rightarrow\) \textbf{LEGO approach}

Building a tracking code is an \textbf{entirely local problem}.
Each element is defined by a local Hamiltonian.
LEGO Approach: Defining Blocks

- Two main families of blocks can be defined.
  - (a.) **Rectilinear block** consisting of two parallel entrance and exit faces separated by a distance $L$.
    → useful for straight section, quadrupole, sextupole magnets
  - (b.) **Curvilinear block** where the vertical axes are parallel but the horizontal are tilted by an angle $\theta$. We have then $L = \rho \theta$.
    → suitable for a dipole magnet of curvature radius $\rho$ deviating the particle trajectory by approximatively $\theta$ degrees.

Transfer Application or Flow: $\vec{x}^f = M_{i \rightarrow f} \vec{x}^i$

- For each block, we need to construct an application mapping the entrance coordinates to the exit coordinates, which we call resp. initial (i) and final (f) coordinates.
- This application is called the transfer map or the flow of the system:
  $$\vec{x}^f = M_{i \rightarrow f} \vec{x}^i$$

Remarks:
1. A magnetic element can be defined as a **collection of such a block**.
2. A physical element is defined as set of blocks (rectilinear or curvilinear) and a model defining the transfer map.
Properties of Transfer Maps (i)

Law of composition

- Let be two elements (1) and (2)
- Let be $M_{0 \rightarrow 1}$ and $M_{1 \rightarrow 2}$ their transfer maps
- Then the transfer map of the system (1–2) is given by the composition of both maps:

$$M_{0 \rightarrow 2} = M_{1 \rightarrow 2} M_{0 \rightarrow 1}$$ (4)

Transfer map of the full ring and stability

- Let be an autonomous system modeling a ring
- Its transfer map is then the composition of flow of all $m$ elements of ring of circumference $2\pi R$:

$$A_{2\pi R} = M_{(m-1) \rightarrow m} \ldots M_{0 \rightarrow 1}$$ (5)

Properties of Transfer Maps (ii)

Transfer map and stability

- Moreover, the map over $n$-turns is:

$$A_{2n\pi R \rightarrow 2(n-1)\pi R} \ldots A_{2\pi R \rightarrow 4\pi R} A_{0 \rightarrow 2\pi R} = (A_{0 \rightarrow 2\pi R})^n$$ (6)

- The stability is given by the iteration of this map over initial condition $\vec{x}^i$
- This is indeed a local approach where each element is modeled using a Hamiltonian expressed in the most suitable coordinates.

A complex problem of integration has been broken out in a few simple ones

How to compute the map of an individual element modeled by an Hamiltonian $H$? numerical integration is a solution.
Integrator Properties

Tracking over many turns (thousands for lepton machines, billions for hadron machines). This requires:

- **Very good stability** of the numerical integrator (no spurious damping, no divergence)
- **Few error accumulations** (systematic and method errors): controled or bounded errors
- **High precision** (order of the integrator) to reduce the the number of steps whence the time of computation.
- **Conservation of the Hamiltonian**, *i.e.* the energy of the system (**symplecticity**)
- Conservation of the angular momentum, n-form, . . .

What classes of integrators are available?
Integration Classes

Runge Kutta Schemes
- Order 3, 4
- 4/5 with adaptative stepsize

Predicator/Corrector Methods
Adams’ schemes
These are rather slow numerical integration methods

Symplectic Integrators
- Split operator symplectic integrator
- Monomial full integrability
- Energy conservation

Building Symplectic Schemes

Implicit methods
- Implicit Runge–Kutta (Sanz-Serna, 1998)
- Generating functions (Channel and Scovel, 1990)

Explicit methods
- D. Ruth first introduced in 1983 based on successive canonical transformations
- Neri (1988) extended using Lie Algebra
- Yoshida (1990) give a general scheme for building $2n + 2$ order integrator based on a $2n$ order integrator.
A short History of Symplectic Integration

- **First Introduction**: R. DeVogelaere for FFAGs at MURA project (1956).
- **Big gap for many years ...**
- **Method**: symplectic integrator (Dragt and Finn, 1976)

\[ H_k = A_k + \epsilon B_k \]
where \( A_k \) and \( B_k \) are full integrable

\[ \Rightarrow \text{full ring: } A = \prod_k M_k \]

- **Chosen scheme in this lecture**: Integrators (McLachlan (1998), Laskar and Robutel, 2000)

\[ \Rightarrow \text{symmetric, positive steps, accuracy} \]
Particle Trajectory Computation

Local Hamiltonian Approach

- Motion with respect to a reference particle.
- Canonical variables \((x, p_x), (y, p_y), (l, \delta)\)
- Time-like variable \(s\)

Potential vector \((\hat{A}_x, \hat{A}_y, \hat{A}_s)\)

- Maxwell equations
- Symmetries
- Longitudinal rectangular magnetic profile

\[
\mathcal{H} = -(1 + h(s)x)\sqrt{(1 + \delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2 - e\hat{A}_s} + \delta + 1 \quad (7)
\]

\[
-e\hat{A}_s(x, y) = \text{Re} \left( \sum_{n=1}^{\infty} \frac{b_n + ja_n}{n} (x + jy)^n \right) \quad (8)
\]

with \(h(s) = 1/\rho(s)\).

\(a_n\) and \(b_n\) coefficients: skew and normal 2n–poles
Quick Refresher

- Let $\mathcal{H}$ be the Hamiltonian of a system with $n$ degrees of freedom (DOF)
- Let $\vec{x} = (p_j, q_j)_{j=1..n}$ be generalized positions and momenta and $s$, the independent variable.
- Then the equations of motion are given by the Hamilton equations:

$$\frac{dp_j}{ds} = -\frac{\partial \mathcal{H}}{\partial q_j}, \quad \frac{dq_j}{ds} = \frac{\partial \mathcal{H}}{\partial p_j} \quad \text{for } j = 1..n \quad (9)$$

- Let us define the Poisson bracket of two functions $f$ and $g$ depending on the $\vec{x}$ variable by:

$$\{f, g\} \overset{\text{def}}{=} \sum_j \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \quad (10)$$
Then the equations of the motion can be written as:

$$\frac{d\vec{x}}{ds} = \{\mathcal{H}, \vec{x}\} = L_{\mathcal{H}} \vec{x},$$  \hspace{1cm} (11)

where $L_{\mathcal{H}}$ is a **differential operator** (Lie derivative) defined by $L_{\mathcal{H}} f \overset{\text{def}}{=} \{\mathcal{H}, f\}$.

The equations of motion can then be **formally integrated** for a vector $\vec{x}^i$ of initial conditions:

$$\vec{x}^f = \sum_{n \geq 0} \frac{s^n}{n!} L^n_{\mathcal{H}} \vec{x}^i \overset{\text{def}}{=} e^{sL_{\mathcal{H}}} \vec{x}^i$$ \hspace{1cm} (12)

It is not trivial to determine $e^{sL_{\mathcal{H}}} \vec{x}^i$, since this convergent power series is most of the time slowly convergent. So in practice, one need the evaluation of many terms. Moreover the truncation is not symplectic.

**How to approximate $e^{sL_{\mathcal{H}}}$?**

Mathematically, the previous solution (12) is **symplectic**

The flow transporting the initial conditions to the $s$ position along the system of energy $\mathcal{H}$ preserves the 2-form:

$$dp^f \wedge dq^f = dp^i \wedge dq^i$$ \hspace{1cm} (13)

Any integration method verifying this condition is called an **symplectic integrator**.

A symplectic method conserves the volume (Liouville Theorem, and more generally the Poincaré invariants)

**Remarks:** A Taylor development truncated at order $n$ does not verify most of the time the symplectic condition
In addition, we ask for the energy conservation. If the Hamiltonian is non integrable, such a scheme may not exist in general (Ge and Marsden, 1998).

However this symplectic integrator conserves the Hamiltonian $\tilde{H}$ seen as a perturbed system $\mathcal{H}$

$$\tilde{H} = \mathcal{H} + \sum_{k \geq 1} s^k H_k$$

where $H_k$ is a function of the derivative of order $j \leq k$ of $\mathcal{H}$ (Yoshida, 1990a, 1990b).

Explicit Symplectic Integrators: Construction Principle

Let us write the Hamiltonian as $H = A + \epsilon B$

We look for an approximate Hamiltonian flow $\tilde{x}^f = e^{s\mathcal{L}_A} \tilde{x}^i$ by a product of integrable flows $e^{sL_A}$ and $e^{sL_{\epsilon B}}$.

$L_A$ and $L_{\epsilon B}$ do not commute in general

The Baker-Campbell-Hausdorff (BCH) theorem states

$$e^{sL_A} e^{sL_{\epsilon B}} = e^{s\mathcal{L}_{\tilde{H}}}$$

with the formal Hamiltonian:

$$\tilde{H} = A + \epsilon B + \frac{s}{2} \{ A, \epsilon B \} + \frac{s^2}{12} \left( \{ \{ A, \epsilon B \} \} + \{ \epsilon B, \epsilon B, A \} \right) + \frac{s^3}{24} \{ \{ A, \epsilon B \}, A, \epsilon B \} + \ldots$$
Applications of BCH Theorem

Simplest symplectic integrator of first order is:

\[ e^{sL} \approx e^{sL_A} e^{sL_B} + \mathcal{O}(s\epsilon) \]  

(17)

Using the BCH theorem 15 arbitrary order \( n \) integrators \( (S_n) \) can be built:

\[ S_n(s) = \prod_{i=1}^{n} e^{c_i s L_A} e^{d_i s L_B} \]  

(18)

where the coefficients \( (c_i, d_i)_{i=1..n} \) are found in order to obtain a rest of order \( n: \mathcal{O}(s^n \epsilon) \). It can be easy shown the the two following properties have to be met (exercise):

\[ \sum_{i=1}^{n} c_i = 1 \text{ et } \sum_{i=1}^{n} d_i = 1 \]  

(19)

Symmetric integrators

From now on we study time reversal or symmetric integrators, i.e. \( S_n^{-1}(s) = S_n(-s) \). Show that by construction these integrators are of even rest \( \mathcal{O}(s^{2n} \epsilon) \).

The famous integrator of order 2 leapfrog integrator is written as (Ruth, 1983):

\[ S_2 = e^{c_1 s L_A} e^{d_1 s L_B} e^{c_1 s L_A} \]  

(20)

avec \( c_1 = \frac{1}{2} \) et \( d_1 = 1 \).
Yoshida Scheme

Using the work of Yoshida (1990), a fourth order integrator can be built by composing second order integrators.

\[ S_4(s) = S_2(as)S_2(bs)S_2(as) \] (21)

Noticing that the total step has to be \( s \) and second order term have to be canceled, one gets:

\[
\begin{align*}
2a + b &= 1 \\
2a^3 + b^3 &= 0
\end{align*}
\iff
\begin{align*}
a &= \frac{1}{2 - 2^{\frac{3}{2}}} \\
b &= -\frac{2^{\frac{3}{2}}}{2 - 2^{\frac{3}{2}}}
\end{align*}
\] (22)

Generally Yoshida has proven that a \( 2n + 2 \) order integrator can be built from \( 2n \) order integrators by means of the symplectic scheme:

\[ S_{2n+2}(s) = S_{2n}(as)S_{2n}(bs)S_{2n}(as) \] (23)

One gets easily that : \( a = \frac{1}{2 - 2n + \sqrt{2}} \) et \( b = -\frac{2n + \frac{1}{2}}{2 - 2n + \sqrt{2}} \).

Application to Ruth and Forest Scheme

4th order symplectic integrator (Ruth 1983, Forest and Ruth 1990)

\[ S_4(s) = e^{d_1 sL_A} e^{c_2 sL_B} e^{d_2 sL_A} e^{c_3 sL_B} e^{d_2 sL_A} e^{c_2 sL_B} e^{d_1 sL_A} \] (24)

with \( c_2 = \frac{1}{1 + \alpha}, \ c_3 = (\alpha - 1)c_2, \ d_1 = \frac{c_2}{2}, \ d_2 = \alpha d_1 \), and

\[ \alpha = 1 - 2^{\frac{3}{2}}. \]

(a) \ RUTH
Application to 6th Order Integrator

6th order integrator \( S_6 \) (Yoshida, 1990)

\[
S_6(s) = S_2(ds)S_2(cs)S_2(bs)S_2(as)S_2(bs)S_2(cs)S_2(ds)
\]  (25)

with 3 sets of coefficients (cf. Tab. 1) but always a big negative step

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-0.11776798417887E1</td>
<td>-0.213228522200144E1</td>
</tr>
<tr>
<td>b</td>
<td>+0.235573213359357E0</td>
<td>+0.426068187079180E2</td>
</tr>
<tr>
<td>c</td>
<td>+0.784513610477560E0</td>
<td>+0.143984816797678E1</td>
</tr>
<tr>
<td>d</td>
<td>1-2(a+b+c)</td>
<td></td>
</tr>
</tbody>
</table>

Table: Three sets of coefficients for building a 6th order symplectic integrator using Yoshida method. In every case, there is a big negative step since Eq. 19 has to be satisfied: \( d + 2(a + b + c) = 1 \).

Negative steps give numerical instabilities: in practice we never go beyond 6th order integration.

McLachlan (1995) and Laskar Scheme (2001)

Avoiding negative steps & decreasing the number of terms to evaluate

\( SABA \) and \( SBAB \) Classes

Let us split the Hamiltonian into two parts \( A \) et \( B \), symmetric symplectic integrators can be obtained from one of the two classes \( SABA_k \) and \( SBAB_k \):

\[
\begin{align*}
SABA_{2n} & : e^{c_1 s_{LA}} e^{d_1 s_{LB}} \ldots e^{c_{n+1} s_{LA}} e^{d_n s_{LB}} \ldots e^{d_1 s_{LB}} e^{c_1 s_{LA}} \\
SABA_{2n+1} & : e^{c_1 s_{LA}} e^{d_1 s_{LB}} \ldots e^{c_{n+1} s_{LA}} e^{d_n s_{LB}} e^{c_n s_{LA}} \ldots e^{d_1 s_{LB}} e^{c_1 s_{LA}} \\
SBAB_{2n} & : e^{d_1 s_{LB}} e^{c_2 s_{LA}} e^{d_2 s_{LB}} \ldots e^{c_{n+1} s_{LA}} e^{d_n s_{LB}} e^{c_n s_{LA}} \ldots e^{d_2 s_{LB}} e^{c_2 s_{LA}} e^{d_1 s_{LB}} \\
SBAB_{2n+1} & : e^{d_1 s_{LB}} e^{c_2 s_{LA}} e^{d_2 s_{LB}} \ldots e^{c_{n+1} s_{LA}} e^{d_n s_{LB}} e^{c_n s_{LA}} \ldots e^{d_2 s_{LB}} e^{c_2 s_{LA}} e^{d_1 s_{LB}}
\end{align*}
\]  (26)

NB: \( k \) is the number of evaluations of \( e^{c_k s_{LA}} (e^{d_k s_{LB}}) \) operators.
First Applications

- For example the second order leapfrog integrator (Eq. 20) belongs to the SABA₁: \( \tilde{H} = A + \epsilon B + O(s^2\epsilon) \).
- the 4th order Forest and Ruth integrator (Eq. 24) belongs to SABA₃ with \( \tilde{H} = A + \epsilon B + O(s^4\epsilon) \).

Two negative steps in this method give absolute value larger than 1: \( d_1 \approx 0.6756 \), \( d_2 \approx -0.1756 \), \( c_2 \approx 1.3512 \) et \( c_3 \approx -1.7024 \).

For large integration steps, lost of efficiency (high cost, num. instab.)

- Suzuki (1991) demonstrated the impossibility to build up an symplectic integrator or order higher that \( n > 2 \) with only positive steps.
- Nevertheless this problem can be partly solved
  Indeed the small parameter \( \epsilon \) has not been exploited up to now.
- Solution: computing \((c_j, d_j)\) of the integrators Eq. 26 to get a rest of order \( O(s^n\epsilon + s^2\epsilon^2) \) instead of \( O(s^n\epsilon) \).

Second Order Class

Example of 4th order integrator SABA₂:

\[
SABA₂ = e^{c₁sLA}e^{d₁sLB}e^{c₂sLA}e^{d₁sLB}e^{c₁sLA}
\]  

(27)

with a unique solution for the coefficients:

\[
d₁ = \frac{1}{2}, \quad c₁ = \frac{1}{2}(1 - c₂), \quad \text{and}, \quad c₂ = \frac{1}{\sqrt{3}} \quad \text{with}
\]

\[
\tilde{H} = A + \epsilon B + s²\epsilon² \left( -\frac{1}{24} + \frac{c₁}{4} \right) \{\{A, B\}, B\} + O(s⁴\epsilon)
\]

(28)

Similarly for integrator SBAB₂, on gets,

\[
SBAB₂ = e^{d₁sLB}e^{c₂sLA}e^{d₂sLB}e^{c₂sLA}e^{d₁sLB}
\]

(29)

with the unique triplet:

\[
d₁ = \frac{1}{8}, \quad d₂ = \frac{2}{3} \quad \text{et} \quad c₂ = \frac{1}{2}
\]
Improvement

If $A$ is quadratic in the momenta and $B$ depends only of the positions, the scheme can be improved by introduction a corrector term $C$ (Laskar and Robutel, 2000):

$$ C = e^{-s^3 \epsilon^2 \frac{c}{2} L_{\{(A,B),B\}}} $$

(30)

where $c$ is chosen to cancel the term of order $O(s^2 \epsilon^2)$. 

**Remark:** the corrector term introduces a negative step which decreases with the order of the integration scheme.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$SABA_n$</th>
<th>$SBAB_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/12</td>
<td>-1/24</td>
</tr>
<tr>
<td>2</td>
<td>$(2 - \sqrt{3})/24$</td>
<td>1/72</td>
</tr>
</tbody>
</table>

Table: Coefficient $c$ for the corrector for $SABA_n$ et $SBABA_n$ (Laskar and Robutel, 2000)

Application

Typical symplectic scheme for $SABA_2$ integrator:

$$ SABA_{AC} = e^{-s^3 \epsilon^2 \frac{c}{2} L_{\{(A,B),B\}}} SABA_2 e^{-s^3 \epsilon^2 \frac{c}{2} L_{\{(A,B),B\}}} $$

(31)

the integrator is still symmetric with a rest of order $O(s^n \epsilon + s^4 \epsilon^2)$.

In theory, the **Yoshida scheme** can be used to build an integrator of order $2n + 2$ using an integrator of order $2n$, with:

$$ S_{2n+2}(s) = S_{2n}(s) S_{2n}(cs) S_{2n}(s) $$

(32)

by choosing the $c$ term in such a way that the $2n$ order term is canceled i.e. $c = -2^{\frac{1}{2n+1}}$. In practice, this new integrator has a step $2 + c$, with $c \approx 1$. Nevertheless the cost is around **three times more expensive** than that the cost of the $S_{2n}$ integrator.
Bounded Error of Symplectic Integrators

Remarkable Properties

- High stability
- Slow phase shift with time
- Error is bounded

Application for a combined dipole

4th order integrators

Forest and Ruth (1990)

$S_{ABAC_2}$ and $S_{ABAC_2}$
Local Hamiltonian Approach

- Motion with respect to a reference particle.
- Canonical variables $(x, p_x), (y, p_y), (l, \delta)$
- Time-like variable $s$

### Potential vector $(\hat{A}_x, \hat{A}_y, \hat{A}_s)$

- Maxwell equations
- Symmetries
- Longitudinal rectangular magnetic profile

\[ \mathcal{H} = -(1 + h(s)x)\sqrt{(1 + \delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2 - e\hat{A}_s + \delta + 1} \]  

\[ -e\hat{A}_s(x, y) = \text{Re} \left( \sum_{n=1}^{\infty} \frac{b_n + j a_n}{n} (x + jy)^n \right) \]

with $h(s) = 1/\rho(s)$.

$a_n$ and $b_n$ coefficients: skew and normal 2n–poles
Straight Section: Definition & Equation

- **Element of length** $L$ **without magnetic field** ($e\hat{A}_s = 0$).
- **Linear** equation: the cinematic term is approximated at first order (**paraxial or small angle approximation**).
- Hamiltonian in **rectangular coordinates** ($h = 0$) (cf. Eq. 7)

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = \frac{p_x^2 + p_y^2}{2(1 + \delta)} \quad (33)$$

- $(x, y, l)$ are **cyclic coordinates** so their conjugated momentum is a first integral of the motion

$$\begin{align*}
\frac{dx}{ds} &= \frac{p_x}{1 + \delta} \\
\frac{dy}{ds} &= \frac{p_y}{1 + \delta} \\
\frac{dl}{ds} &= -\frac{p_x^2 + p_y^2}{2(1 + \delta)^2} \\
\end{align*} \quad \begin{align*}
\frac{dp_x}{ds} &= 0 \\
\frac{dp_y}{ds} &= 0 \\
\frac{d\delta}{ds} &= 0
\end{align*} \quad (34)$$

Straight Section: Transfer Map

The differential system 34 is **fully integrable**

$$\begin{align*}
x'^{i} &= x^{i} + \frac{p_x^{i}}{1 + \delta} s \\
y'^{i} &= y^{i} + \frac{p_y^{i}}{1 + \delta} s \\
l'^{i} &= l^{i} - \frac{(p_x^{i})^2 + (p_y^{i})^2}{2(1 + \delta)^2} s
\end{align*} \quad \begin{align*}
p_x'^{i} &= p_x^{i} \\
p_y'^{i} &= p_y^{i} \\
\delta'^{i} &= \delta
\end{align*} \quad (35)$$

where the $i$ et $f$ are the canonical coordinates respectively at entrance and exit of the straight section of $L = s$. 

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Dipole Magnets: Definition & Equations

- Element with a curvature radius \( \rho_c \) and a length \( L \)
- On average it curves the particle trajectory by an angle \( \theta = L/\rho_c \)
- Assumption: \( \rho_c = \rho = h^{-1} \)
- Potential for a dipole magnet: \(-e\hat{A}_s = h(s)\left(x + \frac{x^2}{2\rho_c}\right)\)
- **Quadratic term** in \( x \) means a horizontal purely geometric focusing.
- Hamiltonian in **curvilinear coordinates** using Eq. 7:

\[
\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1+hx)\frac{p_x^2 + p_y^2}{2(1+\delta)} - hx(1+\delta) + h\left(x + \frac{x^2}{2\rho_c}\right)
\]

Combined Dipole: General case

- Hamiltonian \( \mathcal{H}(x, y, l, p_x, p_y, \delta) \) is derived from equations 7 and 8:

\[
\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta)} + hx\frac{p_x^2 + p_y^2}{2(1+\delta)} - hx(1+\delta) + h(x + h\frac{x^2}{2}) + \frac{b_2}{2}(x^2 - y^2)
\]

\[
= \frac{p_x^2 + p_y^2}{2(1+\delta)} + h\delta x + h^2\frac{x^2}{2} + \frac{b_2}{2}(x^2 - y^2)
\]

(37)

with \( b_2 \) the gradient term

- **Term of small machine**: often neglected (e.g. BETA, TRACY) except in MAD (Iselin, 1985b) **source of chromaticities**.
Combined dipole: Reduction of the Hamiltonian

- **Approximations**: large rings, small angle, and hard edge
- Reduction of the Hamiltonian to:

\[
\mathcal{H}(x, y, l, p_x, p_y, \delta) = \frac{p_x^2 + p_y^2}{2(1 + \delta)} - h\delta x + \frac{h^2 x^2}{2} + \frac{b_2}{2}(x^2 - y^2)
\]

(38)

- Equations of motion:

\[
\begin{align*}
\frac{dx}{ds} &= \frac{p_x}{1 + \delta} \\
\frac{dy}{ds} &= \frac{p_y}{1 + \delta} \\
\frac{dl}{ds} &= -\frac{p_x^2 + p_y^2}{2(1 + \delta)^2} - hx
\end{align*}
\]

\[
\begin{align*}
\frac{dp_x}{ds} &= h\delta - (h^2 + b_2)x \\
\frac{dp_y}{ds} &= b_2 y \\
\frac{d\delta}{ds} &= 0
\end{align*}
\]

(39)

Combined Dipole: Transfer Map

- Equations 39 are **fully integrable**.
- We prefer to use the previous symplectic integration scheme (more stable numerical solutions).
- Hamiltonian split **into 2 parts**: A (straight section or drift) and B (multipole or kick). Sometimes called kick–drift method.

\[
\mathcal{H} = A + B \quad \text{with} \quad A = \frac{p_x^2 + p_y^2}{2(1 + \delta)} \quad \text{and} \quad B = -h\delta x + \frac{h^2 x^2}{2} + \frac{b_2}{2}(x^2 - y^2)
\]

(40)

- Conditions for using an integrator of class \(SABA_n\) or \(SBAB_n\)
- Computation is eased up: **only one Poisson bracket** to compute!

\[
e^{SLA} : \begin{cases} x^f = x^i + \frac{p_x^i}{1 + \delta} s \\ y^f = y^i + \frac{p_y^i}{1 + \delta} s \\ l^f = l^i - \frac{(p_x^i)^2 + (p_y^i)^2}{2(1 + \delta)^2} s - h x^i s - h \frac{p_x^i}{2} s^2 \end{cases} \quad \begin{cases} p_x^f = p_x^i \\ p_y^f = p_y^i \\ \delta^f = \delta \end{cases}
\]

(41)
Combined Dipole: Transfer Map (ii)

- Transfer map for B part

\[ e^{sL_B} : \begin{cases} 
x'(i) = x^i \\
y'(i) = y^i \\
l'(i) = l^i \\
p_x'(i) = p_x^i - ((b_2 + h^2)x^i - h\delta) s \\
p_y'(i) = p_y^i + b_2 y^i s \\
\delta'(i) = \delta 
\end{cases} \tag{42} \]

- Improvement of the integration scheme with adding a corrector term given by the double Poisson bracket

\[ \{\{A, B\}, B\} = \frac{1}{1 + \delta} \left((\alpha + kx)^2 + b_2^2 y^2\right) \tag{43} \]

with \(\alpha = -\delta h\) et \(k = b_2 + h^2\).

- Corrector transfer map:

\[ e^{sL_{\{\{A, B\}, B\}}} : \begin{cases} 
x'(i) = x^i \\
y'(i) = y^i \\
P_x'(i) = P_x^i - \frac{2k(\alpha + kx^i)}{1 + \delta} s \\
P_y'(i) = P_y^i - \frac{2b_2^2 y^i}{1 + \delta} s \\
\delta'(i) = \delta 
\end{cases} \tag{44} \]

with \(s = -s^3 \frac{c}{2}\) cf. Eq. 30.

Quadrupole Magnet: Definition & Equation

- Element of length \(L\) and gradient \(b_2 = K\) focusing the beam in on plane and defocusing it in the other one.

- Small angle approximation and hard-edge model

- Hamiltonian in **rectangular coordinates** 7 et 8 :

\[ H(x, y, l, p_x, p_y, \delta) = \frac{p_x^2 + p_y^2}{2(1 + \delta)} + \frac{K}{2}(x^2 - y^2) \tag{45} \]

- Equation of motion are (**l is cyclic**) :

\[ \begin{cases} 
\frac{dx}{ds} = \frac{p_x}{1 + \delta} \\
\frac{dy}{ds} = \frac{p_y}{1 + \delta} \\
\frac{dl}{ds} = -\frac{p_x^2 + p_y^2}{2(1 + \delta)^2} \\
\frac{dp_x}{ds} = -Kx \\
\frac{dp_y}{ds} = Ky \\
\frac{d\delta}{ds} = 0 
\end{cases} \tag{46} \]
Quadrupole Magnet: Direct Integration

- Direct integration for $K > 0$:

\[
\begin{align*}
x' &= \cos(\omega s)x' + \frac{1}{\omega(1+\delta)} \sin(\omega s)p'_x \\
y' &= \cosh(\omega s)y' + \frac{1}{\omega(1+\delta)} \sinh(\omega s)p'_y \\
l' &= l' + \Delta l
\end{align*}
\]

with $\omega = \sqrt{\frac{K}{1+\delta}}$, $s = L$ and

\[
\Delta l = \frac{1}{4} \omega \left( \frac{1}{2} \sin(2\omega s) - \omega s \right) (x')^2 - \frac{1}{4} \omega \left( \frac{1}{2} \sinh(2\omega s) - \omega s \right) (y')^2
\]

\[
- \frac{1}{4} \frac{1}{(1+\delta)^2} \left( \frac{\sin(2\omega s)}{2\omega} + s \right) (p'_x)^2 - \frac{1}{4} \frac{1}{(1+\delta)^2} \left( \frac{\sinh(2\omega s)}{2\omega} + s \right) (p'_y)^2
\]

\[
\frac{1}{2} \frac{\sin^2(\omega s)}{1+\delta} x'p'_x - \frac{1}{2} \frac{\sinh^2(\omega s)}{1+\delta} y'p'_y
\]

Multipole Magnets

- Sextupole magnet of length $L$ and strength $b_3 = S$
- Small machine approximation and hard-edge model.
- Hamiltonian in rectangular coordinates Eq. 7 and 8

\[
\mathcal{H}(x, y, l, p_x, p_y, \delta) = \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{S}{3} (x^3 - 3xy^2)
\]

- Equations of the motion ($l$ est cyclique):

\[
\begin{align*}
\frac{dx}{ds} &= \frac{p_x}{1+\delta} \\
\frac{dy}{ds} &= \frac{p_y}{1+\delta} \\
\frac{dl}{ds} &= -\frac{p_x^2 + p_y^2}{2(1+\delta)^2} \\
\frac{dp_x}{ds} &= -S(x^2 - y^2) \\
\frac{dp_y}{ds} &= 2Sxy \\
\frac{d\delta}{ds} &= 0
\end{align*}
\]
Generalization to 2n-poles

- 2-n pole of length $L$ and strength $b_n$

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = \frac{p_x^2 + p_y^2}{2(1 + \delta)} + \frac{b_n}{n} \Re \left( (x + jy)^n \right)$$

- $A(p_x, p_y, \delta)$ is the Hamiltonian of a straight section
- Integration of $B(x, y)$

$$e^{sL_B} x^i : \begin{cases} x^f = x^i \\ y^f = y^i \\ l^f = l^i \end{cases} \quad \begin{cases} p_x^f = p_x^i - b_n \Re(x + jy)^{n-1} s \\ p_y^f = p_y^i + b_n \Im(x + jy)^{n-1} s \\ \delta^f = \delta \end{cases}$$

with $s = L$. 

Outline

- Introduction
- Numerical integration
- Integration Methods
- Symplectic Integration ab Ovo
- Refresher on Hamiltonian Mechanics
- Application to Single Element
- Conclusion
- References
Conclusion

- General methods have been given to integrate the equations of motion governed by a Hamiltonian system.
- **Symplectic integrators** are the best candidate when space phase volume and energy have to be conserved.
- Symplectic integrators have **remarkable properties**. They are well adapted for **long term integration** (tracking over many turns, computation of the motion of celestial bodies over million of years ...).
- **Their error is bounded** and its size depends of the integrator order chosen.
- They are **easy to construct and of use, cost effective**.
- Examples of tracking through a few few standard magnets have been given.
- The next step is to have a **refined Fourier analysis tool** for exploring the (beam) nonlinear dynamics.

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